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On an algorithmic construction of lump solutions in a 2+1 integrable equation

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Abstract
The singular manifold method is used to generate lump solutions of a generalized integrable nonlinear Schrödinger equation in 2+1 dimensions. We present several essentially different types of lump solutions. The connection between this method and the Ablowitz–Villarroel scheme is also analysed.

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1. Introduction

In this paper, we shall consider the following (2 + 1)-dimensional nonlinear system:

$$m_y + u\omega = 0 \qquad \mathrm{i}u_t + u_{xx} + 2um_x = 0 \qquad -\mathrm{i}\omega_t + \omega_{xx} + 2\omega m_x = 0, \tag{1.1}$$

where $u(x, y, t)$, $w(x, y, t)$ and $m(x, y, t)$ are three fields.

There are several relevant considerations that motivate interest in this system.

- We note that ω can be taken to be the complex conjugate of u. If, in addition, u and m are assumed to satisfy

$$\lim_{x^2+y^2\to\infty} u(x, y) = \lim_{y\to-\infty} m(x, y) = 0, \tag{1.2}$$

  we recover the fundamental equation proposed first by Shulman [1] and later embedded by Fokas [2] into a more general, parameter-dependent, family of equations. Further reduction to the manifold $x = y$ yields the nonlinear Schrödinger equation. Equation (1.1) is also discussed in [3–5].
- It possesses the Painlevé property (cf [6, 7]).
- Line soliton and dromion solutions were obtained in [4, 7] by use of the Hirota method and the singular manifold method (SMM), respectively.
- Equation (1.1) can be mapped into the generalized dispersive wave equation by means of a Miura transformation [8, 9].

- Darboux transformations for this equation are considered in [4], using the singular manifold method.
- The system is integrable since it arises as the compatibility of a Lax pair (see section 3). The spectral theory of the relevant spatial operator (cf (3.13)) under certain nonvanishing boundary conditions has been considered in [8] in connection with another interesting wave equation (and hence with a modified temporal evolution). It has been shown that under these boundary conditions only a continuous spectrum may arise. However, a discrete spectrum also exists and is related to the appearance of lumps. A description of this spectrum will be addressed in a future paper.

This paper is devoted to determining an interesting family of regular and rationally decaying solutions—in short the lumps—to the above equation. Although the analysis of [8] precludes the existence of rapidly decaying localized configurations under the boundary conditions considered, we show that (1.1) supports the existence of weakly decaying potentials. To this end, we construct an algorithmic procedure based on the singular manifold method of [10]. No connection with the inverse scattering transformation will be attempted.

Lump configurations are paradigmatic solutions of integrable equations in 2+1 dimensions and as such they have been extensively studied in recent years. They were first found with direct methods in [11] in the context of the prototype integrable equation in 2+1 dimensions: the KPI equation. The role that they play from the spectral point of view was highlighted in [12] (see also [13]). In [14], it was shown that DSII also supports rationally decaying solutions. Since then, lumps have appeared in many other integrable equations; see, for example, [15] for the three-dimensional sine-Gordon, which contains an expansion of the eigenfunctions in terms of poles that strongly suggests a connection with the Painlevé methods. An interesting feature of these objects is the triviality of their dynamics: the motion is uniform and, upon scattering, the multilump solution regains its properties and does not even undergo a displacement in its position. However, for the KPI equation a new class of localized, real-valued solutions with rational decay has recently been found that exhibit nontrivial asymptotic dynamics. In this regard see [16, 17], in which a study of the connection with the discrete spectrum of the relevant spectral problem is described. A subsequent study via direct methods appears in [18]. These solutions exhibit interesting scattering properties that were first noticed in [19]. The extension of these ideas and solutions to DSII equation via spectral analysis of the Dirac operator on the plane is considered in [20], while a complete study based on direct methods is performed in [21], see also [22] for related ideas. For some interacting solutions in the Yang–Mills equation framework, see [23].

We next briefly review the contents of this paper:

- Following [4] with the appropriate notation, in section 2 we recall the main results of the singular manifold method for (1.1).
- Section 3 is devoted to using SMM as a procedure that allows us to iterate fields, the eigenfunctions and the singular manifolds in such a way that we can iteratively obtain new solutions arising from trivial seed solutions. Darboux transformations of the Lax pair appear in a very natural way in this scheme.
- In section 4, the above method is used to generate several basic lump solutions of (1.1) with different properties.
- Two-lump solutions are obtained explicitly in section 5, using the same iteration procedure.
- Expansion in poles of the eigenfunctions is considered in section 6. The connection between the SMM method and the results of [17] is indicated but will be the object of future research.

2. The singular manifold method

The SMM method has proved to be an effective tool for clarifying several properties of the nonlinear partial differential equations that have the Painlevé property; in particular, it has been used to obtain Lax pairs [24], soliton and dromion solutions [24] and Miura [9, 25] and Darboux transformations [26]. Remarkably, the SMM seems to be more successful in $2+1$ dimensions than in the $1+1$ case (cf [25]).

In this section, we shall apply the SMM to equation (1.1). A similar analysis has also been carried out in [9], but here we shall present an improved and more convenient version of the results. Nevertheless, for a more comprehensive explanation of how to proceed on applying the SMM to (1.1), we address the reader to [9].

2.1. Truncated expansion

Let $\phi^{(0)}(x, y, t) = 0$ be a singular manifold depending of the initial conditions. As is well known [10], the SMM implies that the fields can be expanded as a truncated Painlevé series of the form

$$\begin{aligned} u^{(1)} &= u^{(0)} + A^{(0)} \frac{\phi_x^{(0)}}{\phi^{(0)}} \\ \omega^{(1)} &= \omega^{(0)} + B^{(0)} \frac{\phi_x^{(0)}}{\phi^{(0)}} \\ m^{(1)} &= m^{(0)} + \frac{\phi_x^{(0)}}{\phi^{(0)}}, \end{aligned} \tag{2.1}$$

where the notation means that $\{m^{(0)}, u^{(0)}, \omega^{(0)}\}$ is a seed solution and $\phi^{(0)}(x, y, t)$ is the singular manifold corresponding to this solution. $A^{(0)}(x, y, t)$ and $B^{(0)}(x, y, t)$ are the functions to be determined by substitution of (2.1) into (1.1). $\{m^{(1)}, u^{(1)}, \omega^{(1)}\}$ is a new solution of (1.1) obtained through the auto-Bäcklund transformation (2.1). In what follows, the index (0) will correspond to the seed solution and (1) to the first iteration. In general, the index (n) will correspond to the nth iteration of the auto-Bäcklund transformation (2.1) and, consequently, $\phi^{(n)}$ will denote the singular manifold corresponding to the $\{m^{(n)}, u^{(n)}, \omega^{(n)}\}$ solution.

2.2. Definitions

For every integer n, we find that it is useful to define the following quantities:

$$v^{(n)} = \frac{\phi_{xx}^{(n)}}{\phi_x^{(n)}}, \qquad q^{(n)} = \frac{\phi_y^{(n)}}{\phi_x^{(n)}}, \qquad r^{(n)} = \frac{\phi_t^{(n)}}{\phi_x^{(n)}}, \tag{2.2}$$

as well as the Schwarzian derivative $s^{(n)}$:

$$s^{(n)} = v_x^{(n)} - \frac{(v^{(n)})^2}{2}. \tag{2.3}$$

From the above definitions, the following relations arise immediately:

$$r_y^{(n)} - q_t^{(n)} + r^{(n)} q_x^{(n)} - q^{(n)} r_x^{(n)} = 0 \tag{2.4}$$

$$v_y^{(n)} = \left(q_x^{(n)} + q^{(n)} v^{(n)}\right)_x \tag{2.5}$$

$$v_t^{(n)} = \left(r_x^{(n)} + r^{(n)} v^{(n)}\right)_x. \tag{2.6}$$

2.3. Seed solutions and singular manifold equations

By direct substitution of (2.1) in (1.1), one obtains three different polynomials in $\phi^{(0)}$. By setting each coefficient of these polynomials to 0 and by using (2.2)–(2.6) we obtain several equations that relate $v^{(0)}, q^{(0)}, r^{(0)}, A^{(0)}, B^{(0)}$ with $\{m^{(0)}, u^{(0)}, \omega^{(0)}\}$. They can be used to determine the invariants $s^{(0)}, q^{(0)}, r^{(0)}$ in terms of $A^{(0)}, B^{(0)}$. This is most conveniently done using MAPLE software. The following results are obtained:

(a) $s^{(0)}, q^{(0)}, r^{(0)}, A^{(0)}, B^{(0)}$ satisfy the following relations:

$$q^{(0)} = A^{(0)} B^{(0)} \tag{2.7}$$

$$r_x^{(0)} = \frac{1}{2}\left(\frac{\mathrm{i}A_{xx}^{(0)}}{A^{(0)}} - \mathrm{i}\frac{B_{xx}^{(0)}}{B^{(0)}} - \frac{A_t^{(0)}}{A^{(0)}} - \frac{B_t^{(0)}}{B^{(0)}}\right) \tag{2.8}$$

$$s^{(0)} = -\frac{A_{xx}^{(0)}}{A^{(0)}} - \frac{B_{xx}^{(0)}}{B^{(0)}} - \mathrm{i}\frac{A_t^{(0)}}{A^{(0)}} + \mathrm{i}\frac{B_t^{(0)}}{B^{(0)}} - \frac{(r^{(0)})^2}{2} + \int r_t^{(0)}\,\mathrm{d}x. \tag{2.9}$$

Equations (2.4)–(2.9) constitute the singular manifold equations.

(b) $\{m^{(0)}, u^{(0)}, \omega^{(0)}\}$ can be written in terms of the singular manifold in the following way:

$$u^{(0)} = -\frac{A^{(0)}}{2}\left(\mathrm{i}r^{(0)} + v^{(0)} + \frac{A_x^{(0)}}{A^{(0)}}\right) \tag{2.10}$$

$$\omega^{(0)} = -\frac{B^{(0)}}{2}\left(-\mathrm{i}r^{(0)} + v^{(0)} + \frac{B_x^{(0)}}{B^{(0)}}\right) \tag{2.11}$$

$$m_x^{(0)} = -\frac{1}{4}\left(2v_x^{(0)} + \frac{A_{xx}^{(0)}}{A^{(0)}} + \frac{B_{xx}^{(0)}}{B^{(0)}} + \mathrm{i}\frac{A_t^{(0)}}{A^{(0)}} - \mathrm{i}\frac{B_t^{(0)}}{B^{(0)}}\right). \tag{2.12}$$

As we shall prove in the next section, these expressions—(2.10)–(2.12)—for the seed solutions can be linearized, giving rise to the Lax pair for the system (1.1).

3. The Lax pair

(a) Definition of the eigenfunctions

To linearize the above equations (see [4]), we introduce two functions $\psi^{(0)}$ and $\varphi^{(0)}$ via

$$v^{(0)} = \frac{\psi_x^{(0)}}{\psi^{(0)}} + \frac{\varphi_x^{(0)}}{\varphi^{(0)}} \tag{3.1}$$

$$r^{(0)} = \mathrm{i}\left(\frac{\psi_x^{(0)}}{\psi^{(0)}} - \frac{\varphi_x^{(0)}}{\varphi^{(0)}}\right). \tag{3.2}$$

Substitution of (3.1) and (3.2) into (2.8) and (2.9) gives us

$$A_t^{(0)} = \mathrm{i}\frac{A^{(0)}}{2}\left(2\frac{A_{xx}^{(0)}}{A^{(0)}} - \frac{\psi_{xx}^{(0)}}{\psi^{(0)}} + 3\frac{\varphi_{xx}^{(0)}}{\varphi^{(0)}} - 4\frac{\left(\varphi_x^{(0)}\right)^2}{(\varphi^{(0)})^2} - \mathrm{i}\frac{\psi_t^{(0)}}{\psi^{(0)}} + \mathrm{i}\frac{\varphi_t^{(0)}}{\varphi^{(0)}}\right) \tag{3.3}$$

$$B_t^{(0)} = -\mathrm{i}\frac{B^{(0)}}{2}\left(2\frac{B_{xx}^{(0)}}{B^{(0)}} - \frac{\varphi_{xx}^{(0)}}{\varphi^{(0)}} + 3\frac{\psi_{xx}^{(0)}}{\psi^{(0)}} - 4\frac{\left(\psi_x^{(0)}\right)^2}{(\psi^{(0)})^2} - \mathrm{i}\frac{\psi_t^{(0)}}{\psi^{(0)}} + \mathrm{i}\frac{\varphi_t^{(0)}}{\varphi^{(0)}}\right). \tag{3.4}$$

Substitution of (3.1)–(3.4) into (2.4) and (2.5) yields (after integration in x)

$$A_x^{(0)} = \frac{1}{B^{(0)}}\frac{\psi_y^{(0)}}{\psi^{(0)}} - A^{(0)}\frac{\varphi_x^{(0)}}{\varphi^{(0)}} \tag{3.5}$$

$$B_x^{(0)} = \frac{1}{A^{(0)}}\frac{\varphi_y^{(0)}}{\varphi^{(0)}} - B^{(0)}\frac{\psi_x^{(0)}}{\psi^{(0)}}. \tag{3.6}$$

Finally, substitution of (3.1) and (3.2) into (2.6) yields (after integration in x)

$$\mathrm{i}\frac{\psi_t^{(0)}}{\psi^{(0)}} + \mathrm{i}\frac{\varphi_t^{(0)}}{\varphi^{(0)}} + \frac{\psi_{xx}^{(0)}}{\psi^{(0)}} - \frac{\varphi_{xx}^{(0)}}{\varphi^{(0)}} = 0. \tag{3.7}$$

(b) Seed solution and eigenfunctions

Equations (2.10)–(2.12) for the fields now read

$$u^{(0)} = -\frac{1}{B^{(0)}}\frac{\psi_y^{(0)}}{\psi^{(0)}} \tag{3.8}$$

$$\omega^{(0)} = -\frac{1}{A^{(0)}}\frac{\varphi_y^{(0)}}{\varphi^{(0)}} \tag{3.9}$$

$$4m_x^{(0)} = -\mathrm{i}\frac{\psi_t^{(0)}}{\psi^{(0)}} + \mathrm{i}\frac{\varphi_t^{(0)}}{\varphi^{(0)}} - \frac{\psi_{xx}^{(0)}}{\psi^{(0)}} - \frac{\varphi_{xx}^{(0)}}{\varphi^{(0)}}. \tag{3.10}$$

(c) Spatial part of the Lax pair

Solving (3.8) and (3.9) for $A^{(0)}$ and $B^{(0)}$, we have

$$A^{(0)} = -\frac{1}{\omega^{(0)}}\frac{\varphi_y^{(0)}}{\varphi^{(0)}} \tag{3.11}$$

$$B^{(0)} = -\frac{1}{u^{(0)}}\frac{\psi_y^{(0)}}{\psi^{(0)}}, \tag{3.12}$$

whose substitution into (3.5) and (3.6) provides

$$u^{(0)}\psi_{xy}^{(0)} - u_x^{(0)}\psi_y^{(0)} - (u^{(0)})^2\omega^{(0)}\psi^{(0)} = 0 \tag{3.13}$$

$$\omega^{(0)}\varphi_{xy}^{(0)} - \omega_x^{(0)}\varphi_y^{(0)} - (\omega^{(0)})^2u^{(0)}\varphi^{(0)} = 0. \tag{3.14}$$

(d) Temporal part of the Lax pair

Adding and subtracting (3.7) and (3.10), we have

$$\mathrm{i}\psi_t^{(0)} + \psi_{xx}^{(0)} + 2m_x^{(0)}\psi^{(0)} = 0 \tag{3.15}$$

$$-\mathrm{i}\varphi_t^{(0)} + \varphi_{xx}^{(0)} + 2m_x^{(0)}\varphi^{(0)} = 0. \tag{3.16}$$

It is trivial to check that the compatibility condition between (3.13)–(3.16) is that $\{u^{(0)}, \omega^{(0)}, m^{(0)}\}$ satisfy (1.1). Actually, (3.13)–(3.16) is a two-component Lax pair for (1.1).

(e) Iteration

Let us recall that by the substitution of (3.11) and (3.12) into (2.1) we obtain

$$
\begin{aligned}
u^{(1)} &= u^{(0)} - \frac{1}{\omega^{(0)}}\frac{\psi^{(0)}\varphi^{(0)}_y}{\phi^{(0)}}\\
\omega^{(1)} &= \omega^{(0)} - \frac{1}{u^{(0)}}\frac{\varphi^{(0)}\psi^{(0)}_y}{\phi^{(0)}}\\
m^{(1)} &= m^{(0)} + \frac{\phi^{(0)}_x}{\phi^{(0)}},
\end{aligned}
\tag{3.17}
$$

where $\psi^{(0)}$ and $\varphi^{(0)}$ are the eigenfunctions of the seed solution $\{u^{(0)}, \omega^{(0)}, m^{(0)}\}$.

(f) Singular manifold and eigenfunctions

According to (2.2)–(2.4), (2.9), (3.1), (3.2), (3.11) and (3.12), $\phi^{(0)}$ can be determined by the eigenfunctions through the exact derivative

$$
\mathrm{d}\phi^{(0)} = \psi^{(0)}\varphi^{(0)}\mathrm{d}x + \frac{1}{u^{(0)}\omega^{(0)}}\psi^{(0)}_y\varphi^{(0)}_y\mathrm{d}y + \mathrm{i}\left(\varphi^{(0)}\psi^{(0)}_x - \psi^{(0)}\varphi^{(0)}_x\right)\mathrm{d}t. \tag{3.18}
$$

4. Darboux transformations

Darboux transformations also arise from our scheme. We follow the procedure developed in [4]. Let $\psi^{(0)}_1, \varphi^{(0)}_1$ and $\psi^{(0)}_2, \varphi^{(0)}_2$ be two different couples of the eigenfunctions for the seed Lax pair such that

$$
\begin{aligned}
&u^{(0)}\left(\psi^{(0)}_j\right)_{xy} - u^{(0)}_x\left(\psi^{(0)}_j\right)_y - (u^{(0)})^2\omega^{(0)}\psi^{(0)}_j = 0\\
&\omega^{(0)}\left(\varphi^{(0)}_j\right)_{xy} - \omega^{(0)}_x\left(\varphi^{(0)}_j\right)_y - (\omega^{(0)})^2u^{(0)}\varphi^{(0)}_j = 0\\
&\mathrm{i}\left(\psi^{(0)}_j\right)_t + \left(\psi^{(0)}_j\right)_{xx} + 2m^{(0)}_x\psi^{(0)}_j = 0\\
&-\mathrm{i}\left(\varphi^{(0)}_j\right)_t + \left(\varphi^{(0)}_j\right)_{xx} + 2m^{(0)}_x\varphi^{(0)}_j = 0, \qquad j = 1, 2.
\end{aligned}
\tag{4.1}
$$

Equation (3.18) allows us to construct two zero-order eigenfunctions—$\phi^{(0)}_1, \phi^{(0)}_2$—through the expression

$$
\mathrm{d}\phi^{(0)}_j = \psi^{(0)}_j\varphi^{(0)}_j\mathrm{d}x + \frac{1}{u^{(0)}\omega^{(0)}}\left(\psi^{(0)}_j\right)_y\left(\varphi^{(0)}_j\right)_y\mathrm{d}y + \mathrm{i}\left[\varphi^{(0)}_j\left(\psi^{(0)}_j\right)_x - \psi^{(0)}_j\left(\varphi^{(0)}_j\right)_x\right]\mathrm{d}t,\quad j = 1, 2 \tag{4.2}
$$

and by using $\psi^{(0)}_1, \varphi^{(0)}_1$ in (3.17) we have an iterated solution

$$
\begin{aligned}
u^{(1)} &= u^{(0)} - \frac{1}{\omega^{(0)}}\frac{\psi^{(0)}_1\left(\varphi^{(0)}_1\right)_y}{\phi^{(0)}_1}\\
\omega^{(1)} &= \omega^{(0)} - \frac{1}{u^{(0)}}\frac{\varphi^{(0)}_1\left(\psi^{(0)}_1\right)_y}{\phi^{(0)}_1}\\
m^{(1)} &= m^{(0)} + \frac{\left(\phi^{(0)}_1\right)_x}{\phi^{(0)}_1}.
\end{aligned}
\tag{4.3}
$$

Equation (4.3) suggests an iteration of the eigenfunctions of the form (see [4, 26])

$$\begin{aligned}\psi_2^{(1)} &= \psi_2^{(0)} - \psi_1^{(0)}\frac{\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}\\ \varphi_2^{(1)} &= \varphi_2^{(0)} - \varphi_1^{(0)}\frac{\Delta_{1,2}^{(0)}}{\phi_1^{(0)}},\end{aligned} \tag{4.4}$$

such that $\psi_2^{(1)}$ and $\varphi_2^{(1)}$ are the eigenfunctions for the iterated solution $\{u^{(1)}, \omega^{(1)}, m^{(1)}\}$ obtained by the Painlevé expansion of the seed eigenfunctions $\psi_2^{(0)}$ and $\varphi_2^{(0)}$. This means that they satisfy the Lax pair:

$$\begin{aligned}&u^{(1)}\big(\psi_2^{(1)}\big)_{xy} - u_x^{(1)}\big(\psi_2^{(1)}\big)_y - (u^{(1)})^2\omega^{(1)}\psi_2^{(1)} = 0\\ &\omega^{(1)}\big(\varphi_2^{(1)}\big)_{xy} - \omega_x^{(1)}\big(\varphi_2^{(1)}\big)_y - (\omega^{(1)})^2u^{(1)}\varphi_2^{(1)} = 0\\ &\mathrm{i}\big(\psi_2^{(1)}\big)_t + \big(\psi_2^{(1)}\big)_{xx} + 2m_x^{(1)}\psi_2^{(1)} = 0\\ &-\mathrm{i}\big(\varphi_2^{(1)}\big)_t + \big(\varphi_2^{(1)}\big)_{xx} + 2m_x^{(1)}\varphi_2^{(1)} = 0.\end{aligned} \tag{4.5}$$

Furthermore, the singular manifold itself can be iterated as

$$\phi_2^{(1)} = \phi_2^{(0)} - \frac{\Omega_{1,2}^{(0)}\Delta_{1,2}^{(0)}}{\phi_1^{(0)}}, \tag{4.6}$$

such that $\phi_2^{(1)}$ is the singular manifold that arises from $\psi_2^{(1)}$ and $\varphi_2^{(1)}$ through the following expression:

$$\mathrm{d}\phi_2^{(1)} = \psi_2^{(1)}\varphi_2^{(1)}\mathrm{d}x + \frac{\big(\psi_2^{(1)}\big)_y\big(\varphi_2^{(1)}\big)_y}{u^{(1)}\omega^{(1)}}\mathrm{d}y + \mathrm{i}\big[\varphi_2^{(1)}\big(\psi_2^{(1)}\big)_x - \psi_2^{(1)}\big(\varphi_2^{(1)}\big)_x\big]\,\mathrm{d}t. \tag{4.7}$$

Substitution of (4.3) and (4.4) into (4.5) gives us the following expressions for $\Omega_{i,j}^{(0)}$ and $\Delta_{ij}^{(0)}$:

$$\begin{aligned}&\mathrm{d}\Omega_{i,j}^{(0)} = \varphi_i^{(0)}\psi_j^{(0)}\mathrm{d}x + \frac{\big(\varphi_i^{(0)}\big)_y\big(\psi_j^{(0)}\big)_y}{u^{(0)}\omega^{(0)}}\,\mathrm{d}y + \mathrm{i}\big[\varphi_i^{(0)}\big(\psi_j^{(0)}\big)_x - \big(\varphi_i^{(0)}\big)_x\psi_j^{(0)}\big]\,\mathrm{d}t\\ &\Delta_{i,j}^{(0)} = \Omega_{j,i}^{(0)},\\ &i, j = 1, 2.\end{aligned} \tag{4.8}$$

These results are consistent with the substitution of (4.6) into (4.7). It is interesting to note that the matrix $\Omega_{i,j}^{(0)}$ defined in (4.8) allows us to write (4.3) as

$$\phi_j^{(0)} = \Omega_{j,j}^{(0)}. \tag{4.9}$$

Summarizing, equations (4.3) and (4.4) can be considered a binary Darboux transformation that transform Lax pair (4.1) into Lax pair (4.5).

5. Iteration

The Darboux transformations obtained in the previous section can be applied several times in order to construct an iteration procedure that will allow us to obtain more and more complicated solutions arising from a trivial seed. Equation (4.3) can therefore be considered as the first of these iterations. Let us now iterate the procedure again.

5.1. Second iteration

Let $\phi_2^{(1)}$ define a singular manifold for the iterated solution $\{u^{(1)}, \omega^{(1)}, m^{(1)}\}$. We can therefore iterate again (4.3) to obtain

$$
\begin{aligned}
u^{(2)} &= u^{(1)} - \frac{1}{\omega^{(1)}}\frac{\psi_2^{(1)}\left(\varphi_2^{(1)}\right)_y}{\phi_2^{(1)}}\\
\omega^{(2)} &= \omega^{(1)} - \frac{1}{u^{(1)}}\frac{\varphi_2^{(1)}\left(\psi_2^{(1)}\right)_y}{\phi_2^{(1)}}\\
m^{(2)} &= m^{(1)} + \frac{\left(\phi_2^{(1)}\right)_x}{\phi_2^{(1)}}.
\end{aligned}
\tag{5.1}
$$

Combining this and (4.6)–(4.9) we obtain a second iterated solution

$$
\begin{aligned}
u^{(2)} &= u^{(0)} - \frac{1}{\omega^{(0)}}\left[\left(\varphi_1^{(0)}\right)_y\left(\psi_1^{(0)}\phi_2^{(0)} - \psi_2^{(0)}\Omega_{2,1}^{(0)}\right) + \left(\varphi_2^{(0)}\right)_y\left(\psi_2^{(0)}\phi_1^{(0)} - \psi_1^{(0)}\Omega_{1,2}^{(0)}\right)\right]\frac{1}{\tau_{1,2}}\\
\omega^{(2)} &= \omega^{(0)} - \frac{1}{u^{(0)}}\left[\left(\psi_1^{(0)}\right)_y\left(\varphi_1^{(0)}\phi_2^{(0)} - \varphi_2^{(0)}\Omega_{1,2}^{(0)}\right) + \left(\psi_2^{(0)}\right)_y\left(\varphi_2^{(0)}\phi_1^{(0)} - \varphi_1^{(0)}\Omega_{2,1}^{(0)}\right)\right]\frac{1}{\tau_{1,2}}\\
m^{(2)} &= m^{(0)} + \frac{(\tau_{1,2})_x}{\tau_{1,2}},
\end{aligned}
\tag{5.2}
$$

where

$$
\tau_{1,2} = \phi_2^{(1)}\phi_1^{(0)} = \phi_1^{(0)}\phi_2^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)} = \det\begin{pmatrix}\Omega_{1,1}^{(0)} & \Omega_{1,2}^{(0)}\\ \Omega_{2,1}^{(0)} & \Omega_{2,2}^{(0)}\end{pmatrix}.
\tag{5.3}
$$

Similarly, the matrix element $\Omega_{i,j}^{(0)}$ can be obtained by the iteration of (4.8) as

$$
\mathrm{d}\Omega_{i,j}^{(1)} = \varphi_i^{(1)}\psi_j^{(1)}\,\mathrm{d}x + \frac{\left(\varphi_i^{(1)}\right)_y\left(\psi_j^{(1)}\right)_y}{u^{(1)}\omega^{(1)}}\mathrm{d}y + \mathrm{i}\left[\varphi_i^{(1)}\left(\psi_j^{(1)}\right)_x - \left(\varphi_i^{(1)}\right)_x\psi_j^{(1)}\right]\mathrm{d}t
\tag{5.4}
$$

and by using (4.4) we have

$$
\begin{aligned}
\psi_j^{(1)} &= \psi_j^{(0)} - \psi_1^{(0)}\frac{\Omega_{1,j}^{(0)}}{\phi_1^{(0)}}\\
\varphi_i^{(1)} &= \varphi_i^{(0)} - \varphi_1^{(0)}\frac{\Omega_{i,1}^{(0)}}{\phi_1^{(0)}}.
\end{aligned}
\tag{5.5}
$$

It is easy to check that the truncated Painlevé expansion for the matrix elements is

$$
\Omega_{i,j}^{(1)} = \Omega_{i,j}^{(0)} - \frac{\Omega_{i,1}^{(0)}\Omega_{1,j}^{(0)}}{\phi_1^{(0)}}.
\tag{5.6}
$$

5.2. $(n+1)$th iteration

Obviously, the above procedure can be easily iterated as many times as wished; we can summarize the results for the nth iteration as follows: let $\psi_h^{(n)}$ and $\varphi_h^{(n)}$ be two eigenfunctions of the nth iteration and let $\phi_h^{(n)}$ be the corresponding singular manifold. Then, the iteration scheme is given by

$$u^{(n+1)} = u^{(n)} - \frac{1}{\omega^{(n)}} \frac{\psi_h^{(n)} \left(\varphi_h^{(n)}\right)_y}{\phi_h^{(n)}} \tag{5.7}$$

$$\omega^{(n+1)} = \omega^{(n)} - \frac{1}{u^{(n)}} \frac{\varphi_h^{(n)} \left(\psi_h^{(n)}\right)_y}{\phi_h^{(n)}} \tag{5.8}$$

$$m^{(n+1)} = m^{(n)} + \frac{\left(\phi_h^{(n)}\right)_x}{\phi_h^{(n)}} \tag{5.9}$$

$$\psi_j^{(n+1)} = \psi_j^{(n)} - \psi_h^{(n)} \frac{\Omega_{h,j}^{(n)}}{\phi_h^{(n)}} \tag{5.10}$$

$$\varphi_j^{(n+1)} = \varphi_j^{(n)} - \varphi_h^{(n)} \frac{\Omega_{j,h}^{(n)}}{\phi_h^{(n)}} \tag{5.11}$$

$$\phi_j^{(n+1)} = \phi_j^{(n)} - \frac{\Omega_{j,h}^{(n)} \Omega_{h,j}^{(n)}}{\phi_h^{(n)}} \tag{5.12}$$

$$\Omega_{i,j,}^{(n+1)} = \Omega_{i,j}^{(n)} - \frac{\Omega_{i,h}^{(n)} \Omega_{h,j}^{(n)}}{\phi_h^{(n)}}, \tag{5.13}$$

where $\Omega_{h,j}^{(n)}$ is the matrix defined through the exact derivative

$$\mathrm{d}\Omega_{h,j}^{(n)} = \varphi_h^{(n)} \psi_j^{(n)} \, \mathrm{d}x + \frac{\left(\varphi_h^{(n)}\right)_y \left(\psi_j^{(n)}\right)_y}{u^{(n)} \omega^{(n)}} \mathrm{d}y + \mathrm{i}\left[\varphi_h^{(n)} \left(\psi_j^{(n)}\right)_x - \left(\varphi_h^{(n)}\right)_x \psi_j^{(n)}\right] \mathrm{d}t \tag{5.14}$$

and

$$\phi_j^{(n)} = \Omega_{j,j}^{(n)}. \tag{5.15}$$

In the next section, we shall apply this method to obtain lumps.

6. Lumps

We can now determine certain elementary eigenfunctions $\psi_j^{(0)}, \varphi_j^{(0)}$ by using a trivial seed solution. Let us take the following as the seed solution:

$$u^{(0)} = 1, \qquad \omega^{(0)} = 1, \qquad m^{(0)} = -y.$$

In this case, (4.1) are

$$\begin{aligned} &\left(\psi_j^{(0)}\right)_{xy} - \psi_j^{(0)} = 0 \\ &\left(\varphi_j^{(0)}\right)_{xy} - \varphi_j^{(0)} = 0 \\ &\mathrm{i}\left(\psi_j^{(0)}\right)_t + \left(\psi_j^{(0)}\right)_{xx} = 0 \\ &-\mathrm{i}\left(\varphi_j^{(0)}\right)_t + \left(\varphi_j^{(0)}\right)_{xx} = 0. \end{aligned} \tag{6.1}$$

Elementary solutions of (6.1) are

$$\begin{aligned} &\psi_j^{(0)} = \mathrm{e}^{k_j Q(x,y,t,k_j)} P^{[N]}(x, y, t, k_j), \\ &\varphi_j^{(0)} = \mathrm{e}^{-n_j Q(x,y,t,n_j)} P^{[M]}(x, y, -t, -n_j), \\ &j = 1, 2, \end{aligned} \tag{6.2}$$

where k_j and n_j are complex constants, and $Q(k_j)$ is

$$Q(x,y,t,k_j) = x + \frac{y}{k_j^2} + \mathrm{i}k_j t. \tag{6.3}$$

$P^{[N]}(x,y,t,k_j)$ is a polynomial of degree N in x of the form

$$P^{[N]}(k_j) = x^N + \alpha_1^{[N]}(y,t,k_j)x^{N-1} + \cdots + \alpha_{N-1}^{[N]}(y,t,k_j)x + \alpha_N^{[N]}(y,t,k_j), \qquad N \geqslant 0, \tag{6.4}$$

whose coefficients $\alpha[N]_h$ are the functions of y and t that are defined by the recursion relation

$$\begin{aligned}
\frac{\partial \alpha_{h+1}^{[N]}}{\partial y} &= \frac{(h-N)}{k_j^2}\left[\alpha_h^{[N]} + k_j \frac{\partial \alpha_h^{[N]}}{\partial y}\right] \\
\frac{\partial \alpha_{h+1}^{[N]}}{\partial t} &= \mathrm{i}(h-N)\left[\alpha_{h-1}^{[N]}(h-N-1) - 2k_j\alpha_h^{[N]}\right] \\
h &= 0,\ldots,N-1, \qquad \alpha^{[0]} = 1.
\end{aligned} \tag{6.5}$$

The first four polynomials are, for instance,

$$\begin{aligned}
P^{[0]}(x,y,t,k_j) &= 1 \\
P^{[1]}(x,y,t,k_j) &= x + 2\mathrm{i}k_j t - \frac{y}{k_j^2} \\
P^{[2]}(x,y,t,k_j) &= x^2 + 2\left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)x + \left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)^2 + 2\mathrm{i}t + \frac{2y}{k_j^3} \\
P^{[3]}(x,y,t,k_j) &= x^3 + 3\left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)x^2 + 3\left(\left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)^2 + 2\mathrm{i}t + \frac{2y}{k_j^3}\right)x \\
&\quad + \left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)^3 + 3\left(2\mathrm{i}k_j t - \frac{y}{k_j^2}\right)\left(2\mathrm{i}t + \frac{2y}{k_j^3}\right) - 6\frac{y}{k_j^4}.
\end{aligned} \tag{6.6}$$

If the potential is to be a rational solution, as happens with the lumps, one expects that the singular manifold must reduce to a polynomial. From (4.2) and (6.3), it is clear that for this to happen we need to select $n_j = k_j$. Furthermore, it is easy to check (see (6.3)–(6.5)) that $Q^*(x,y,t,k_j) = Q(x,y,t,-k_j^*)$ and $P^{[N]}(x,y,-t,k_j^*) = (P^{[N]}(x,y,t,k_j))^*$. This suggests that we should select $k_2 = -k_1^*$. Summarizing, in the following we have

$$\begin{aligned}
&n_1 = k_1, \qquad n_2 = k_2 = -k_1^*, \\
&\psi_1^{(0)} = \mathrm{e}^{k_1 Q_1} P^{[N]}(x,y,t,k_1), \qquad \varphi_1^{(0)} = \mathrm{e}^{-k_1 Q_1} P^{[M]}(x,y,-t,-k_1), \\
&\psi_2^{(0)} = \mathrm{e}^{-k_1^* Q_1^*} P^{[M]}(x,y,t,-k_1^*), \qquad \varphi_2^{(0)} = \mathrm{e}^{k_1^* Q_1^*} P^{[N]}(x,y,-t,k_1^*),
\end{aligned} \tag{6.7}$$

where $Q_1 = Q(k_1)$. We can classify the different solutions according to the different values of N and M.

6.1. Lumps of type 0+0

These can be obtained by setting $N = M = 0$. In this case, (6.7) yields

$$P^{[0]}(k_1) = 1, \qquad P^{[0]}(-k_1^*) = 1. \tag{6.8}$$

These expressions for the eigenfunctions allow us to compute the matrix $\Omega^{(0)}_{i,j}$ defined in (4.8):

$$\begin{aligned} d\phi_1^{(0)} &= d\Omega_{1,1}^{(0)} = dx - \frac{1}{k_1^2}\,dy + 2ik_1\,dt,\\ d\phi_2^{(0)} &= d\Omega_{2,2}^{(0)} = dx - \frac{1}{(k_1^*)^2}\,dy - 2ik_1^*\,dt,\\ d\Omega_{1,2}^{(0)} &= \left[dx + \frac{1}{k_1k_1^*}\,dy + i(k_1-k_1^*)\,dt\right]e^{-k_1Q_1}\,e^{-k_1^*Q_1^*},\\ d\Omega_{2,1}^{(0)} &= \left[dx + \frac{1}{k_1k_1^*}\,dy + i(k_1-k_1^*)\,dt\right]e^{k_1Q_1}\,e^{k_1^*Q_1^*}, \end{aligned} \tag{6.9}$$

which can be integrated as

$$\begin{aligned} \Omega_{1,1}^{(0)} &= \phi_1^{(0)} = x - \frac{y}{k_1^2} + 2ik_1t = X_1 + iY_1,\\ \Omega_{2,2}^{(0)} &= \phi_2^{(0)} = x - \frac{y}{(k_1^*)^2} - 2ik_1^*t = X_1 - iY_1,\\ \Omega_{1,2}^{(0)} &= \frac{-1}{k_1+k_1^*}\,e^{-k_1Q_1}\,e^{-k_1^*Q_1^*} = \frac{-1}{2a_1}\,e^{-k_1Q_1}\,e^{-k_1^*Q_1^*},\\ \Omega_{2,1}^{(0)} &= \frac{1}{k_1+k_1^*}\,e^{k_1Q_1}\,e^{k_1^*Q_1^*} = \frac{1}{2a_1}\,e^{k_1Q_1}\,e^{k_1^*Q_1^*}, \end{aligned} \tag{6.10}$$

where we have

$$k_1 = a_1 + ib_1 \tag{6.11}$$

$$X_1 = \frac{P_1+P_1^*}{2} = x - \frac{\left(a_1^2-b_1^2\right)}{\left(a_1^2+b_1^2\right)^2}y - 2b_1t \tag{6.12}$$

$$Y_1 = \frac{P_1-P_1^*}{2i} = \frac{2a_1b_1}{\left(a_1^2+b_1^2\right)^2}y + 2a_1t. \tag{6.13}$$

Therefore, the function $\tau_{1,2}$ can be written as the following real positive defined expression:

$$\tau_{1,2} = X_1^2 + Y_1^2 + \left(\frac{1}{2a_1}\right)^2. \tag{6.14}$$

In this case, (5.2) provides the solution for the second iteration:

$$\begin{aligned} u^{(2)} &= 1 - \frac{1}{\tau_{1,2}}\,\frac{1+2i(b_1X_1+a_1Y_1)}{a_1^2+b_1^2},\\ \omega^{(2)} &= 1 - \frac{1}{\tau_{1,2}}\,\frac{1-2i(b_1X_1+a_1Y_1)}{a_1^2+b_1^2},\\ m^{(2)} &= -y + \frac{(\tau_{1,2})_x}{\tau_{1,2}}, \end{aligned} \tag{6.15}$$

where $\tau_{1,2}$ is given in (6.14). Figure 1 represents the behaviour of $m_y^{(2)}$.

6.2. Lumps of type 0+1

These can be obtained by setting $N = 0$, $M = 1$. Accordingly, we have

$$\begin{aligned} \psi_1^{(0)} &= e^{k_1Q_1}, \qquad \varphi_1 = e^{-k_1Q_1}P^{[1]}\\ \psi_2^{(0)} &= e^{-k_1^*Q_1^*}(P^{[1]})^*, \qquad \varphi_2^{(0)} = e^{k_1^*Q_1^*}, \end{aligned} \tag{6.16}$$

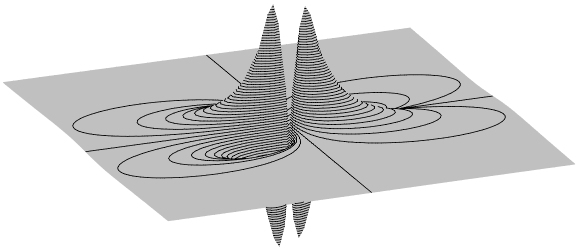

Figure 1. Lump of type 0+0: $m_y^{(2)}$.

where

$$P^{[1]} = x + 2\mathrm{i}k_1 t - \frac{y}{k_1^2}. \tag{6.17}$$

By integrating (4.8), and with the aid of (6.11)–(6.14), we have

$$\begin{aligned}\Omega_{1,1}^{(0)} &= \phi_1^{(0)} \\ &= \frac{1}{2}\left[X_1^2 - Y_1^2 - \frac{a_1^2 - 3b_1^2}{a_1^2 + b_1^2}\frac{(Y_1 - 2a_1 t)}{b_1}\right] + \mathrm{i}\left[X_1 Y_1 - t + \frac{3a_1^2 - b_1^2}{a_1^2 + b_1^2}\frac{(Y_1 - 2a_1 t)}{2a_1}\right]\\ \Omega_{2,2}^{(0)} &= \phi_2^{(0)} = \left(\phi_1^{(0)}\right)^* \end{aligned} \tag{6.18}$$

$$\Omega_{1,2}^{(0)} = -\frac{1}{2a_1}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right]\frac{1}{\mathrm{e}^{k_1 Q_1}\,\mathrm{e}^{k_1^* Q_1^*}} \tag{6.19}$$

$$\Omega_{2,1}^{(0)} = \frac{1}{2a_1}\mathrm{e}^{k_1 Q_1}\,\mathrm{e}^{k_1^* Q_1^*}. \tag{6.20}$$

By substituting (6.18)–(6.20) into (5.3), we have the positive defined expression for $\tau_{1,2}$:

$$\begin{aligned}\tau_{1,2} = \frac{1}{4}&\left[X_1^2 - Y_1^2 - \frac{a_1^2 - 3b_1^2}{a_1^2 + b_1^2}\frac{(Y_1 - 2a_1 t)}{b_1}\right]^2 + \left[X_1 Y_1 - t + \frac{3a_1^2 - b_1^2}{a_1^2 + b_1^2}\frac{(Y_1 - 2a_1 t)}{2a_1}\right]^2\\ &+ \frac{1}{4a_1^2}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right].\end{aligned} \tag{6.21}$$

Figure 2 shows the field $m_y^{(2)}$ corresponding to this lump solution.

6.3. *Lumps of type 1+1*

These can be obtained by setting $N = M = 1$. In this case, the seed eigenfunctions are

$$\psi_1^{(0)} = \mathrm{e}^{k_1 Q_1} P^{[1]}, \qquad \varphi_1^{(0)} = \mathrm{e}^{-k_1 Q_1} P^{[1]} \tag{6.22}$$

$$\psi_2^{(0)} = \mathrm{e}^{-k_1^* Q_1^*}(P^{[1]})^*, \qquad \varphi_2^{(0)} = \mathrm{e}^{k_1^* Q_1^*}(P^{[1]})^*. \tag{6.23}$$

By integrating (4.9), we have

$$\begin{aligned}\Omega_{1,1}^{(0)} = \phi_1^{(0)} = &\left[\frac{X_1^3}{3} - X_1 Y_1^2 + \frac{\left(a_1^4 - 6a_1^2 b_1^2 + b_1^4\right)}{2a_1 b_1\left(a_1^2 + b_1^2\right)^2}(Y_1 - 2a_1 t)\right]\\ &+ \mathrm{i}\left[-\frac{Y_1^3}{3} + X_1^2 Y_1 - \frac{2\left(a_1^2 - b_1^2\right)}{\left(a_1^2 + b_1^2\right)^2}(Y_1 - 2a_1 t)\right]\end{aligned}$$

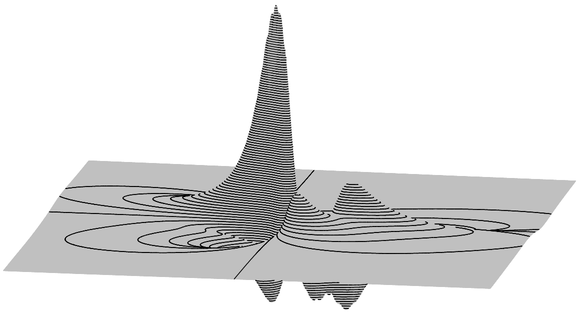

Figure 2. Lump of type 0+1: $m_y^{(2)}$.

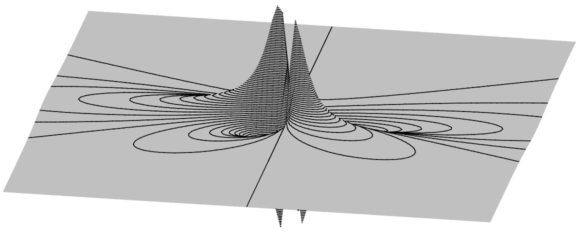

Figure 3. Lump of type 1+1: $m_y^{(2)}$.

$$\Omega_{2,2}^{(0)} = \phi_2^{(0)} = \left(\phi_1^{(0)}\right)^*$$
$$\Omega_{1,2}^{(0)} = -\frac{1}{2a_1}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right] \mathrm{e}^{-k_1 Q_1}\,\mathrm{e}^{-k_1^* Q_1^*}$$
$$\Omega_{2,1}^{(0)} = \frac{1}{2a_1}\left[\left(X_1 - \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right] \mathrm{e}^{k_1 Q_1}\,\mathrm{e}^{k_1^* Q_1^*}. \tag{6.24}$$

By substituting (6.24) into (5.3) and by using (6.12)–(6.14) we have the positive defined expression for $\tau_{1,2}$:

$$\tau_{1,2} = \left[\frac{X_1^3}{3} - X_1 Y_1^2 + \frac{\left(a_1^4 - 6a_1^2 b_1^2 + b_1^4\right)}{2a_1 b_1 \left(a_1^2 + b_1^2\right)^2}(Y_1 - 2a_1 t)\right]^2 + \left[-\frac{Y_1^3}{3} + X_1^2 Y_1 - \frac{2\left(a_1^2 - b_1^2\right)}{\left(a_1^2 + b_1^2\right)^2}(Y_1 - 2a_1 t)\right]^2 + \frac{1}{4a_1^2}\left[\left(X_1^2 + Y_1^2\right)^2 + \frac{Y_1^2}{a_1^2} + \frac{1}{4a_1^4}\right]. \tag{6.25}$$

7. Two-lump solution

In a previous paper [4], we saw that the one-soliton solution corresponds to the first iteration of a seed solution, while the two-soliton solution arises from the second iteration in such a way that each iteration provides a different wave number. Nevertheless, as we have seen in section 6 obtaining a one-lump solution requires two iterations, such that the second one provides a wave number that is the complex conjugate of the first one. This suggests that the two-lump solution requires four iterations, such that the first and the third introduce two wave numbers k_1 and k_3, while the second and the fourth give us the complex conjugate $k_2 = -k_1^*$ and $k_4 = -k_3^*$.

To clarify the procedure, let us consider $\phi_1^{(0)}, \phi_2^{(1)}, \phi_3^{(2)}, \phi_4^{(3)}$, where the superindex indicates the order of the iteration and the subindex refers to the wave number related to the eigenfunction. According to (5.12), we can write the following chain of iterations:

$$\begin{aligned}\phi_2^{(1)} &= \phi_2^{(0)} - \frac{\Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}}{\phi_1^{(0)}}\\ \phi_3^{(2)} &= \phi_3^{(1)} - \frac{\Omega_{2,3}^{(1)}\Omega_{3,2}^{(1)}}{\phi_2^{(1)}}\\ \phi_4^{(3)} &= \phi_4^{(2)} - \frac{\Omega_{3,4}^{(2)}\Omega_{4,3}^{(2)}}{\phi_3^{(2)}},\end{aligned} \tag{7.1}$$

and (5.13) gives us

$$\Omega_{i,j}^{(1)} = \Omega_{i,j}^{(0)} - \frac{\Omega_{i,1}^{(0)}\Omega_{1,j}^{(0)}}{\phi_1^{(0)}} \qquad \Omega_{i,j}^{(2)} = \Omega_{i,j}^{(1)} - \frac{\Omega_{i,2}^{(1)}\Omega_{2,j}^{(1)}}{\phi_2^{(1)}}. \tag{7.2}$$

From (5.10) and (5.11), we have

$$\begin{aligned}\psi_j^{(1)} &= \psi_j^{(0)} - \psi_1^{(0)}\frac{\Omega_{1,j}^{(0)}}{\phi_1^{(0)}}, \qquad \psi_j^{(2)} = \psi_j^{(1)} - \psi_2^{(1)}\frac{\Omega_{2,j}^{(1)}}{\phi_2^{(1)}}\\ \varphi_j^{(1)} &= \varphi_j^{(0)} - \varphi_1^{(0)}\frac{\Omega_{j,1}^{(0)}}{\phi_1^{(0)}}, \qquad \varphi_j^{(2)} = \varphi_j^{(1)} - \varphi_2^{(1)}\frac{\Omega_{j,2}^{(1)}}{\phi_2^{(1)}}.\end{aligned} \tag{7.3}$$

The same iteration can be applied to the fields. In particular, for m we have

$$m^{(3)} = m^{(2)} + \frac{\left(\phi_3^{(2)}\right)_x}{\phi_3^{(2)}} \qquad m^{(4)} = m^{(3)} + \frac{\left(\phi_4^{(3)}\right)_x}{\phi_4^{(3)}},$$

which, combined with (4.6) and (5.1), can be written as

$$m^{(4)} = m^{(0)} + \frac{(\tau_{(1,2,3,4)})_x}{\tau_{(1,2,3,4)}}, \tag{7.4}$$

where

$$\tau_{(1,2,3,4)} = \phi_4^{(3)}\phi_3^{(2)}\phi_2^{(1)}\phi_1^{(0)}. \tag{7.5}$$

Equation (7.2) can be used in (7.5) to give

$$\tau_{(1,2,3,4)} = \left(\phi_4^{(2)}\phi_3^{(2)} - \Omega_{3,4}^{(2)}\Omega_{4,3}^{(2)}\right)\left(\phi_2^{(0)}\phi_1^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}\right). \tag{7.6}$$

Employing (7.1) and (7.2) in (7.6) again, we have

$$\begin{aligned}\tau_{(1,2,3,4)} &= \left(\phi_2^{(0)}\phi_1^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}\right)\\ &\times\left(\phi_4^1\phi_3^1 - \Omega_{3,4}^{(1)}\Omega_{4,3}^{(1)} + \frac{\Omega_{2,3}^{(1)}\Omega_{3,4}^{(1)}\Omega_{4,2}^{(1)} + \Omega_{2,4}^{(1)}\Omega_{4,3}^{(1)}\Omega_{3,2}^{(1)} - \phi_3^{(1)}\Omega_{2,4}^{(1)}\Omega_{4,2}^{(1)} - \phi_4^{(1)}\Omega_{2,3}^{(1)}\Omega_{3,2}^{(1)}}{\phi_2^{(1)}}\right)\end{aligned} \tag{7.7}$$

or

$$\tau_{(1,2,3,4)} = \left(\phi_2^{(0)}\phi_1^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}\right)\left(\phi_4^{(0)} - \frac{\Omega_{1,4}^{(0)}\Omega_{4,1}^{(0)}}{\phi_1^{(0)}}\right)\left(\phi_3^{(0)} - \frac{\Omega_{1,3}^{(0)}\Omega_{3,1}^{(0)}}{\phi_1^{(0)}}\right)$$
$$-\left(\phi_2^{(0)}\phi_1^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}\right)\left(\Omega_{3,4}^{(0)} - \frac{\Omega_{3,1}^{(0)}\Omega_{1,4}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{4,3}^{(0)} - \frac{\Omega_{4,1}^{(0)}\Omega_{1,3}^{(0)}}{\phi_1^{(0)}}\right)$$
$$+\phi_1^0\left[\left(\Omega_{2,3}^{(0)} - \frac{\Omega_{2,1}^{(0)}\Omega_{1,3}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{3,4}^{(0)} - \frac{\Omega_{3,1}^{(0)}\Omega_{1,4}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{4,2}^{(0)} - \frac{\Omega_{4,1}^{(0)}\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}\right)\right]$$
$$+\phi_1^0\left[\left(\Omega_{2,4}^{(0)} - \frac{\Omega_{2,1}^{(0)}\Omega_{1,4}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{4,3}^{(0)} - \frac{\Omega_{4,1}^{(0)}\Omega_{1,3}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{3,2}^{(0)} - \frac{\Omega_{3,1}^{(0)}\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}\right)\right]$$
$$-\phi_1^0\left[\left(\phi_3^{(0)} - \frac{\Omega_{3,1}^{(0)}\Omega_{1,3}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{2,4}^{(0)} - \frac{\Omega_{2,1}^{(0)}\Omega_{1,4}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{4,2}^{(0)} - \frac{\Omega_{4,1}^{(0)}\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}\right)\right]$$
$$-\phi_1^0\left[\left(\phi_4^{(0)} - \frac{\Omega_{4,1}^{(0)}\Omega_{1,4}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{2,3}^{(0)} - \frac{\Omega_{2,1}^{(0)}\Omega_{1,3}^{(0)}}{\phi_1^{(0)}}\right)\left(\Omega_{3,2}^{(0)} - \frac{\Omega_{3,1}^{(0)}\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}\right)\right]. \quad (7.8)$$

By direct calculation in (7.8) it is easy to check that the coefficients in negative powers of $\phi_1^{(0)}$ cancel and the rest of the terms can be collected as

$$\tau_{(1,2,3,4)} = \det\begin{pmatrix} \phi_1^{(0)} & \Omega_{1,2}^{(0)} & \Omega_{1,3}^{(0)} & \Omega_{1,4}^{(0)} \\ \Omega_{2,1}^{(0)} & \phi_2^{(0)} & \Omega_{2,3}^{(0)} & \Omega_{2,4}^{(0)} \\ \Omega_{3,1}^{(0)} & \Omega_{3,2}^{(0)} & \phi_3^{(0)} & \Omega_{3,4}^{(0)} \\ \Omega_{4,1}^{(0)} & \Omega_{4,2}^{(0)} & \Omega_{4,3}^{(0)} & \phi_4^{(0)} \end{pmatrix}. \quad (7.9)$$

Expression (7.9) allows us to obtain the fourth iteration in terms of the solutions of the seed Lax pair. For example, if we use for $\psi_j^{(0)}$, $\varphi_j^{(0)}$ the eigenfunctions of lumps of type 0+0, which means

$$\begin{aligned} \psi_1^{(0)} &= e^{k_1 Q_1} & \varphi_1^{(0)} &= e^{-k_1 Q_1}, & \psi_2^{(0)} &= e^{-k_1^* Q_1^*} & \varphi_2^{(0)} &= e^{k_1^* Q_1^*}, \\ \psi_3^{(0)} &= e^{k_3 Q_3} & \varphi_3^{(0)} &= e^{-k_3 Q_3}, & \psi_4^{(0)} &= e^{-k_3^* Q_3^*} & \varphi_4^{(0)} &= e^{k_3^* Q_3^*}, \end{aligned} \quad (7.10)$$

the determinant (7.9) is, in this case,

$$\tau_{(1,2,3,4)} = \det\begin{pmatrix} P_1 & -\frac{e^{-k_1Q_1}e^{-k_1^*Q_1^*}}{k_1+k_1^*} & \frac{e^{-k_1Q_1}e^{k_3Q_3}}{k_3-k_1} & -\frac{e^{-k_1Q_1}e^{-k_3^*Q_3^*}}{k_3^*+k_1} \\ \frac{e^{k_1Q_1}e^{k_1^*Q_1^*}}{k_1+k_1^*} & P_1^* & \frac{e^{k_1^*Q_1^*}e^{k_3Q_3}}{k_3+k_1^*} & \frac{e^{k_1^*Q_1^*}e^{-k_3^*Q_3^*}}{k_1^*-k_3^*} \\ \frac{e^{-k_3Q_3}e^{k_1Q_1}}{k_1-k_3} & -\frac{e^{-k_1^*Q_1^*}e^{-k_3Q_3}}{k_1+k_3^*} & P_3 & -\frac{e^{-k_3Q_3}e^{-k_3^*Q_3^*}}{k_3+k_3^*} \\ \frac{e^{k_1Q_1}e^{k_3^*Q_3^*}}{k_3^*+k_1} & \frac{e^{k_3^*Q_3^*}e^{-k_1^*Q_1^*}}{k_3^*-k_1^*} & \frac{e^{k_3Q_3}e^{k_3^*Q_3^*}}{k_3+k_3^*} & P_3^* \end{pmatrix}. \quad (7.11)$$

It is easy to see that in the computation of the determinant the exponentials disappear and (7.11) is a real purely rational expression that corresponds to the interaction of two lumps of wave numbers k_1, k_3. Figure 4 shows the behaviour of $m_y^{(4)}$ in this case.

The procedure can be repeated $2m$ times by setting in each iteration $k_{2j} = -k_{2j-1}^*$. Therefore, it is easy to generalize expression (7.9) to the case of m lumps in the following form:

$$\tau_{(2m)} = \left|\Omega_{i,j}^{(0)}\right|, \qquad i, j = 1, \ldots, 2m, \quad (7.12)$$

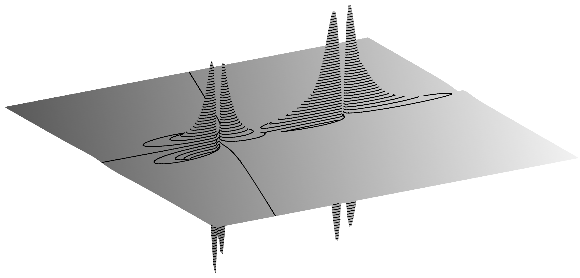

Figure 4. Interaction of two lumps of type I: $m_y^{(4)}$.

where

$$m^{(2m)} = m^{(0)} + \frac{(\tau_{(2m)})_x}{\tau_{(2m)}}.$$

8. On the meromorphic structure of the eigenfunctions of the second iteration

Here we shall explore the analytic structure of the eigenfunctions obtained upon two iterations. Actually, the existence of lump solutions for KPI with nontrivial dynamics has been shown to be related to the possibility of having meromorphic eigenfunctions with higher order poles (see [16, 17]). To this end, in this section we shall obtain the iterated eigenfunctions $\psi_k^{(2)}, \varphi_k^{(2)}$. Let $\phi_1^{(0)}, \phi_2^{(1)}$ be the singular manifolds for the first and second iterations. According to (7.1), we have

$$\phi_2^{(1)} = \phi_2^{(0)} - \frac{\Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)}}{\phi_1^{(0)}} \qquad \phi_k^{(2)} = \phi_k^{(1)} - \frac{\Omega_{2,k}^{(1)}\Omega_{k,2}^{(1)}}{\phi_2^{(1)}}, \tag{8.1}$$

and (5.13) gives us

$$\Omega_{i,j}^{(1)} = \Omega_{i,j}^{(0)} - \frac{\Omega_{i,1}^{(0)}\Omega_{1,j}^{(0)}}{\phi_1^{(0)}} \qquad \Omega_{i,j}^{(2)} = \Omega_{i,j}^{(1)} - \frac{\Omega_{i,2}^{(1)}\Omega_{2,j}^{(1)}}{\phi_2^{(1)}}. \tag{8.2}$$

From (5.10) and (5.11), we have

$$\begin{aligned} \psi_k^{(1)} &= \psi_k^{(0)} - \psi_1^{(0)}\frac{\Omega_{1,k}^{(0)}}{\phi_1^{(0)}}, \qquad \psi_k^{(2)} = \psi_k^{(1)} - \psi_2^{(1)}\frac{\Omega_{2,k}^{(1)}}{\phi_2^{(1)}} \\ \varphi_k^{(1)} &= \varphi_k^{(0)} - \varphi_1^{(0)}\frac{\Omega_{k,1}^{(0)}}{\phi_1^{(0)}}, \qquad \varphi_k^{(2)} = \varphi_k^{(1)} - \varphi_2^{(1)}\frac{\Omega_{k,2}^{(1)}}{\phi_2^{(1)}}. \end{aligned} \tag{8.3}$$

The combination of (8.2) and (8.3) gives us

$$\begin{aligned} \psi_k^{(2)} &= \psi_k^{(0)} - \frac{1}{\tau_{1,2}}\left[\Omega_{1,k}^{(0)}\left(\psi_1^{(0)}\phi_2^{(0)} - \psi_2^{(0)}\Omega_{2,1}^{(0)}\right) + \Omega_{2,k}^{(0)}\left(\psi_2^{(0)}\phi_1^{(0)} - \psi_1^{(0)}\Omega_{1,2}^{(0)}\right)\right] \\ \varphi_k^{(2)} &= \varphi_k^{(0)} - \frac{1}{\tau_{1,2}}\left[\Omega_{k,1}^{(0)}\left(\varphi_1^{(0)}\phi_2^{(0)} - \varphi_2^{(0)}\Omega_{1,2}^{(0)}\right) + \Omega_{k,2}^{(0)}\left(\varphi_2^{(0)}\phi_1^{(0)} - \varphi_1^{(0)}\Omega_{2,1}^{(0)}\right)\right], \end{aligned} \tag{8.4}$$

which allows us to obtain the second iteration $\psi_k^{(2)}, \varphi_k^{(2)}$ of the eigenfunctions through the seed eigenfunctions $\psi_i^{(0)}, \varphi_i^{(0)}$. For simplicity, in the following we shall restrict ourselves to computing $\psi_k^{(2)}$. Obviously, $\varphi_k^{(2)}$ can be obtained in the same way.

In section 2, we computed $\Omega^{(0)}_{i,j}$, $i, j = 1, 2$ for the different types of lumps. In order to have explicit forms for (8.4), we need to calculate $\Omega^{(0)}_{i,j}$, $i, j = 1, 2, k$ in each case, taking into account that according to (8.3) we have

$$\psi^{(0)}_k = e^{kQ_k}[\alpha_k + \beta_k P_k] \tag{8.5}$$

$$P_k = x - \frac{y}{k^2} + 2ikt, \qquad Q_k = x + \frac{y}{k^2} + ikt. \tag{8.6}$$

Therefore, for each type of lump we have two independent possibilities: $\alpha_k = 1$, $\beta_k = 0$ and $\alpha_k = 0$, $\beta_k = 1$. Let us see how to proceed in the case $\alpha_k = 1$, $\beta_k = 0$:

$$\psi^{(0)}_k = e^{kQ_k}. \tag{8.7}$$

8.1. Lumps of type I: two simple poles

The results of section 6.1 give us

$$\begin{aligned}
&\psi^{(0)}_1 = e^{k_1Q_1} \qquad \varphi^{(0)}_1 = e^{-k_1Q_1}, \qquad \psi^{(0)}_2 = e^{-k_1^*Q_1^*} \qquad \varphi^{(0)}_2 = e^{k_1^*Q_1^*},\\
&\phi^{(0)}_1 = P_1, \qquad \phi^{(0)}_2 = P_1^*,\\
&\Omega^{(0)}_{1,2} = -\frac{e^{-k_1Q_1}\, e^{-k_1^*Q_1^*}}{k_1 + k_1^*}, \qquad \Omega^{(0)}_{2,1} = \frac{e^{k_1Q_1}\, e^{k_1^*Q_1^*}}{k_1 + k_1^*},\\
&\tau_{1,2} = P_1P_1^* + \left(\frac{1}{k_1 + k_1^*}\right)^2.
\end{aligned} \tag{8.8}$$

Calculation of $\Omega_{1,k}$ and $\Omega_{2,k}$ by means of (4.9) gives us

$$\Omega_{1,k} = \frac{e^{kQ_k}\, e^{-k_1Q_1}}{k - k_1}, \qquad \Omega_{2,k} = \frac{e^{k_1^*Q_1^*}\, e^{kQ_k}}{k + k_1^*}. \tag{8.9}$$

For $\psi^{(2)}_k$, substitution of (8.6)–(8.8) into (8.4) gives the expression

$$\psi^{(2)}_k = \psi^{(0)}_k\left(1 + \frac{\nu_1}{k - k_1} + \frac{\mu_1}{k + k_1^*}\right), \tag{8.10}$$

where

$$\nu_1 = -\frac{1}{\tau_{1,2}}\left(P_1^* - \frac{1}{k_1 + k_1^*}\right), \qquad \mu_1 = -\frac{1}{\tau_{1,2}}\left(P_1 + \frac{1}{k_1 + k_1^*}\right). \tag{8.11}$$

Expression (8.10) indicates that $\psi^{(2)}_k$ has *two simple poles in* $k = k_1$ *and* $k = -k_1^*$.

8.2. Lumps of type II: a double and a simple pole

The results of section 6.2 give us

$$\begin{aligned}
&\psi^{(0)}_1 = e^{k_1Q_1} \qquad \varphi^{(0)}_1 = e^{-k_1Q_1}P_1, \qquad \psi^{(0)}_2 = e^{-k_1^*Q_1^*}P_1^* \qquad \varphi^{(0)}_2 = e^{k_1^*Q_1^*}\\
&\phi^{(0)}_1 = \frac{P_1^2}{2} - \frac{y}{k_1^3} - it, \qquad \phi^{(0)}_2 = \frac{(P_1^*)^2}{2} - \frac{y}{(k_1^*)^3} + it\\
&\Omega^{(0)}_{1,2} = -e^{-k_1Q_1}e^{-k_1^*Q_1^*}\left(\frac{P_1P_1^*}{k_1 + k_1^*} + \frac{P_1 + P_1^*}{(k_1 + k_1^*)^2} + \frac{2}{(k_1 + k_1^*)^3}\right)\\
&\Omega^{(0)}_{2,1} = \frac{e^{k_1Q_1}\, e^{k_1^*Q_1^*}}{k_1 + k_1^*}\\
&\tau_{1,2} = \left(\frac{P_1^2}{2} - \frac{y}{k_1^3} - it\right)\left(\frac{(P_1^*)^2}{2} - \frac{y}{(k_1^*)^3} + it\right) + \left(\frac{P_1P_1^*}{(k_1 + k_1^*)^2} + \frac{P_1 + P_1^*}{(k_1 + k_1^*)^3} + \frac{2}{(k_1 + k_1^*)^4}\right)
\end{aligned} \tag{8.12}$$

Calculation of $\Omega_{1,k}^{(0)}$ and $\Omega_{2,k}^{(0)}$ by means of (4.9) yields

$$\Omega_{1,k}^{(0)} = \frac{e^{kQ_k}}{e^{k_1Q_1}}\left(\frac{P_1}{k-k_1} - \frac{1}{(k-k_1)^2}\right), \qquad \Omega_{2,k}^{(0)} = \frac{e^{k_1^*Q_1^*}e^{kQ_k}}{k+k_1^*} \tag{8.13}$$

For $\psi_k^{(2)}$, (8.4) gives the expression

$$\psi_k^{(2)} = \psi_k^{(0)}\left(1 + \frac{\nu_2}{(k-k_1)^2} + \frac{\nu_1}{k-k_1} + \frac{\mu_1}{k+k_1^*}\right), \tag{8.14}$$

where

$$\begin{aligned} \nu_2 &= \frac{1}{\tau_{1,2}}\left(\phi_2 - \frac{P_1^*}{k_1+k_1^*}\right), \qquad \nu_1 = -P_1\nu_2 \\ \mu_1 &= -\frac{1}{\tau_{1,2}}\left(P_1^*\phi_1 + \frac{P_1P_1^*}{k_1+k_1^*} + \frac{P_1+P_1^*}{(k_1+k_1^*)^2} + \frac{2}{(k_1+k_1^*)^3}\right). \end{aligned} \tag{8.15}$$

Hence, $\psi_k^{(2)}$ has *a double pole in $k = k_1$ and a single pole in $k = -k_1^*$.*

8.3. Lumps of type III: two double poles

The results of section 6.3 give us

$$\begin{aligned} &\psi_1^{(0)} = e^{k_1Q_1}P_1 \qquad \varphi_1^{(0)} = e^{-k_1Q_1}P_1, \qquad \psi_2^{(0)} = e^{-k_1^*Q_1^*}P_1^* \qquad \varphi_2^{(0)} = e^{k_1^*Q_1^*}P_1^* \\ &\phi_1^{(0)} = \frac{P_1^3}{3} + \frac{y}{k_1^4}, \qquad \phi_2^{(0)} = \frac{(P_1^*)^3}{3} + \frac{y}{(k_1^*)^4} \\ &\Omega_{1,2}^{(0)} = -e^{-k_1Q_1}e^{-k_1^*Q_1^*}\left(\frac{P_1P_1^*}{k_1+k_1^*} + \frac{P_1+P_1^*}{(k_1+k_1^*)^2} + \frac{2}{(k_1+k_1^*)^3}\right) \\ &\Omega_{2,1}^{(0)} = e^{k_1Q_1}e^{k_1^*Q_1^*}\left(\frac{P_1P_1^*}{k_1+k_1^*} - \frac{P_1+P_1^*}{(k_1+k_1^*)^2} + \frac{2}{(k_1+k_1^*)^3}\right) \\ &\tau_{1,2} = \left(\frac{P_1^3}{3} + \frac{y}{k_1^4}\right)\left(\frac{(P_1^*)^3}{3} + \frac{y}{(k_1^*)^4}\right) + \left(\frac{P_1P_1^*}{(k_1+k_1^*)} + \frac{2}{(k_1+k_1^*)^3}\right)^2 - \left(\frac{P_1+P_1^*}{(k_1+k_1^*)^2}\right)^2. \end{aligned} \tag{8.16}$$

Calculation of $\Omega_{1,k}^{(0)}$ and $\Omega_{2,k}^{(0)}$ by means of (4.9) affords

$$\Omega_{1,k}^{(0)} = \frac{e^{kQ}}{e^{k_1Q_1}}\left(\frac{P_1}{k-k_1} - \frac{1}{(k-k_1)^2}\right), \qquad \Omega_{2,k}^{(0)} = e^{k_1^*Q_1^*}e^{k_3Q_3}\left(\frac{P_1^*}{k+k_1^*} - \frac{1}{(k+k_1^*)^2}\right). \tag{8.17}$$

For $\psi_k^{(2)}$, (8.4) gives the expression

$$\psi_k^{(2)} = \psi_k^{(0)}\left(1 + \frac{\nu_2}{(k-k_1)^2} + \frac{\nu_1}{k-k_1} + \frac{\mu_2}{(k+k_1^*)^2} + \frac{\mu_1}{k+k_1^*}\right), \tag{8.18}$$

where we have

$$\begin{aligned} \nu_2 &= \frac{1}{\tau_{1,2}}\left(\phi_2P_1 - P_1^*\left[\frac{P_1P_1^*}{k_1+k_1^*} - \frac{P_1+P_1^*}{(k_1+k_1^*)^2} + \frac{2}{(k_1+k_1^*)^3}\right]\right), \qquad \nu_1 = -P_1\nu_2 \\ \mu_2 &= \frac{1}{\tau_{1,2}}\left(P_1^*\phi_1 + P_1\left[\frac{P_1P_1^*}{k_1+k_1^*} + \frac{P_1+P_1^*}{(k_1+k_1^*)^2} + \frac{2}{(k_1+k_1^*)^3}\right]\right), \qquad \mu_1 = -P_1^*\mu_2. \end{aligned} \tag{8.19}$$

According to (8.18), $\psi_k^{(2)}$ has *two double poles in $k = k_1$ and $k = -k_1^*$.*

The above analysis based on the singular manifold method gives us 'exact' expressions for the iterated eigenfunctions which display an analytic structure with higher order poles and similar to that appearing in [17]. A general study of the analyticity properties of the eigenfunctions belonging to the discrete spectrum will be addressed in future research.

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